

BCS and BEC p -wave pairing in Bose–Fermi gases

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Abstract

The pairing of fermionic atoms in a mixture of atomic fermion and boson gases at zero temperature is investigated. The attractive interaction between fermions, that can be induced by density fluctuations of the bosonic background, can give rise to a superfluid phase in the Fermi component of the mixture. The atoms of both species are assumed to be in only one internal state, so that the pairing of fermions is effective only in odd- l channels. No assumption about the value of the ratio between the Fermi velocity and the sound velocity in the Bose gas is made in the derivation of the energy gap equation. The gap equation is solved without any particular *ansatz* for the pairing field or the effective interaction. The p -wave superfluidity is studied in detail. By increasing the strength and/or decreasing the range of the effective interaction a transition of the fermion pairing regime, from the Bardeen–Cooper–Schrieffer state to a system of tightly bound couples can be realized. These composite bosons behave as a weakly-interacting Bose–Einstein condensate.

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I. INTRODUCTION

Trapped mixtures of ultracold atomic fermions and bosons in gaseous state offer a convenient testing ground for many-body theories [1–4]. These systems display a rich phase diagram. Depending on the strength of the interactions among constituents and on the density of the components, several different phases, like a single mixed phase, a coexistence of pure and mixed phases, or a collapse of the mixture at densities above some critical value, can occur [5–10]. Another interesting feature of these systems is that an effective fermion-fermion interaction, mediated by the bosonic component, can be induced by the exchange of virtual phonons. This fact has been pointed out long ago for dilute solutions of ^3He in superfluid ^4He [11]. The induced interaction becomes particularly relevant in ultracold and diluted gases when the spins of the fermions are polarized by an external magnetic field since in this case the bare interaction between fermions becomes ineffective. This is a consequence of the Pauli principle. An interesting property of the induced interaction between fermions is that it is attractive, irrespective of the sign of the fermion–boson scattering length, and this may lead to the onset of odd- l superfluid phases in the fermion component of the mixture. These fermionic phases would coexist with the bosonic condensate [12–18]. Moreover, as pointed out in [19], results obtained in the study of Bose–Fermi mixtures, could be of interest also in the field of dense QCD systems containing bosonic di-quarks and fermionic unpaired quarks as effective degrees of freedom.

The possibility of controlling the strength of the interatomic interactions via Fano-Feshbach resonances has given further impulse to the study of these systems. After the observation of Fano-Feshbach resonances in Bose-Fermi gas mixtures [20–23], several experimental investigations of the behaviour of these systems as a function of the inter-species interaction strength have been performed [24–27]. On the theoretical side, the study of these systems has addressed mainly the problem of stability against collapse, or that of phase separation with the possible formation of composite fermionic or bosonic molecules. The occurrence of boson-fermion pairing correlations has also been studied [28–36]. Moreover, the authors of Ref. [18] suggest the occurrence of a p -wave superfluid phase in the fermionic component of a Bose-Fermi gas mixture when the inter-species interaction becomes sufficiently strong. They further point out the possibility of a transition from long-ranged pairing to tightly bound pairs of fermions when the interaction strength increases. In Ref. [18], uniform gases in a spin-polarized state are considered, so that the attractive fermion-fermion interaction is provided by the exchange of virtual phonons.

The evolution of superfluidity from the Bardeen, Cooper, Schrieffer (BCS) regime to the Bose-Einstein condensation (BEC) limit has been studied for several different systems of fermions, including ultracold Fermi gases and nuclear matter. In ultracold Fermi gases, the transition occurs when the scattering length changes sign across a Fano-Feshbach resonance [37] (see also [38, 39] for p -wave superfluidity), while in nuclear matter, a BCS-BEC transition is expected when the system becomes rarefied [40–43].

In this work we want to study the transition from the BCS to the BEC regime for the fermionic component of a gaseous mixture of ultracold bosons and spin-polarized fermions. Our treatment is at zero temperature, moreover we assume that the system is sufficiently diluted so that the bare fermion-fermion interaction can be neglected. An attractive interaction between fermions arises due to the exchange of virtual phonons and, because of the Pauli principle, this interaction is effective only for odd values of the relative angular momentum. This induced interaction is derived by extending the results of Ref. [16] to the

limit of zero temperature. In [16] an explicit equation for the energy gap has been derived and solved in the limit in which the sound velocity c_S in the Bose gas is much smaller than the Fermi velocity v_F . Here that approximation is abandoned, since densities and coupling constants are varied over a wide range of values. In Ref. [18] instead, the opposite limit $c_S \gg v_F$ has been assumed, but that approximation is also not appropriate for the same reasons. The obtained equation for the energy gap is solved without introducing any particular *ansatz* for the functional form of the pairing field or the effective interaction. Moreover, the angle-averaging approximation, used in [16], for the quasiparticle energies is dropped. This permits a better knowledge of symmetry properties of the pairing field.

II. FORMALISM

In this paper we rely on the results of Ref. [16]. There, a procedure for determining the induced interaction between fermions has been proposed, by using methods based on quantum field theory at finite temperature.

By treating both components of the mixture in mean-field approximation, the resulting effective interaction was written as

$$V_{eff}(q, \tau_1 - \tau_2) = V_1(q, \tau_1 - \tau_2) + V_2(q, \tau_1 - \tau_2).$$

The first term

$$\begin{aligned} V_1(q, \tau_1 - \tau_2) = & -\lambda^2 n_B^0 \left[D_0^{(11)}(q, \tau_1 - \tau_2) + D_0^{(11)}(q, \tau_2 - \tau_1) \right. \\ & \left. + 2D_0^{(12)}(q, \tau_1 - \tau_2) \right] \end{aligned} \quad (1)$$

is the contribution from states containing one virtual phonon, while the second one

$$\begin{aligned} V_2(q, \tau_1 - \tau_2) = & -\frac{\lambda^2}{(2\pi)^3} \int d\mathbf{k} \left[D_0^{(11)}(k, \tau_1 - \tau_2) D_0^{(11)}(|\mathbf{k} - \mathbf{q}|, \tau_2 - \tau_1) \right. \\ & \left. + D_0^{(12)}(k, \tau_1 - \tau_2) D_0^{(12)}(|\mathbf{k} - \mathbf{q}|, \tau_2 - \tau_1) \right] \end{aligned} \quad (2)$$

is the contribution from states containing two virtual phonons. States with more than two virtual phonons have been neglected.

In the equations above n_B^0 is the density of the Bose condensate, $\lambda = 2\pi a_{BF}/m_R$ is the boson-fermion effective coupling constant, while a_{BF} is the fermion-boson scattering length and $m_R = m_B m_F / (m_B + m_F)$ is the reduced mass for a boson of mass m_B and a fermion of mass m_F . The quantities $D_0^{(ij)}$ are the components of the imaginary-time Bogoliubov propagator (see e.g. sect. 55 of Ref. [46] for their explicit expressions, note however that the definition used here and in [16] differs by an overall minus sign with respect to that of [46]).

The equation for the pairing field at zero temperature can be obtained by analytic continuation to real times ($\tau \rightarrow it$) of Eq. (16) in Ref. [16]. It takes the form

$$\begin{aligned} \Delta(\mathbf{k}, \mathbf{k}', t - t') = & \frac{i}{(2\pi)^3} \int d\mathbf{k}_1 d\mathbf{k}_2 \quad V_{eff}(\mathbf{k}_1 - \mathbf{k}, t - t') \delta(\mathbf{k} + \mathbf{k}' - \mathbf{k}_1 - \mathbf{k}_2) \\ & \times G^{(12)}(\mathbf{k}_2, \mathbf{k}_1, t - t'), \end{aligned} \quad (3)$$

where $G^{(12)}(\mathbf{k}_2, \mathbf{k}_1, t - t')$ is the anomalous propagator for fermions interacting with the pairing field $\Delta(\mathbf{k}, \mathbf{k}', t - t')$ (units $\hbar = c = 1$ are used). The pairing field is antysymmetric under the exchange $\mathbf{k} \leftrightarrow \mathbf{k}'$, this is a consequence of the Pauli principle for a couple of fermions in the same spin state.

For vanishing temperature the contribution to the effective fermion–fermion interaction from the exchange of two phonons can be neglected [16]. The quantum depletion of the boson condensate will also be neglected since, for the purposes of the present work it does not play a significant role. Then the density of the Bose condensate coincides with the actual density of the Bose gas

$$n_B = n_B^0 = \frac{\mu_B - \lambda n_F}{\gamma}. \quad (4)$$

Here $\gamma = 4\pi a_{BB}/m_B$ is the boson–boson effective coupling constant, while a_{BB} is the boson–boson scattering length. Within a mean–field approach the effect of the interaction with the Fermi gas simply amounts to replacing the chemical potential μ_B with the effective value $\mu_B - \lambda n_F$, where n_F is the fermion density [16].

The one–phonon–exchange effective interaction between fermions reads

$$V_1(q, t_1 - t_2) = -\lambda^2 n_B \left[D_0^{(11)}(q, t_1 - t_2) + D_0^{(11)}(q, t_2 - t_1) + 2D_0^{(12)}(q, t_1 - t_2) \right], \quad (5)$$

where $D_0^{(ij)}$ are now the components of the real–time Bogoliubov propagator; for their explicit expressions see e.g. sect. 21 of Ref. [46]. In addition the fermions interact with the condensate mean field acquiring the energy λn_B . This term can be added to the fermionic chemical potential, giving an effective potential $\mu_F^* = \mu_F - \lambda n_B$ to be determined by fixing the density of the Fermi gas.

The solutions of Eq. (3) with center of mass momentum $\mathbf{P} = \mathbf{k} + \mathbf{k}' = \mathbf{k}_1 + \mathbf{k}_2 \neq 0$ correspond to the Larkin–Ovchinnikov–Fulde–Ferrell (LOFF) phase [47]. Here, only solutions with $\mathbf{P} = 0$ are considered. In this case, the pairing field and the anomalous propagator depend on only one momentum, $\mathbf{k} = -\mathbf{k}'$ and $\mathbf{k}_1 = -\mathbf{k}_2$, respectively. In the frequency representation Eq. (3) reads:

$$\Delta(\mathbf{k}, \omega) = \frac{i}{(2\pi)^4} \int d\mathbf{k}' d\omega' V_1(\mathbf{k} - \mathbf{k}', \omega - \omega') G^{(12)}(\mathbf{k}', \omega'). \quad (6)$$

In order to simplify calculations the frequency dependence of the pairing field will be neglected and the static limit $\Delta(\mathbf{k}, \omega) = \Delta(\mathbf{k}, \omega = 0) = \Delta(\mathbf{k})$ will be taken. This approximation amounts to assuming an instantaneous pairing field, neglecting retardation effects for the fermion pairing. In the time–dependent representation:

$$\Delta(\mathbf{k}, t - t') = \delta(t - t') \int d(t - t') \Delta(\mathbf{k}, t - t'). \quad (7)$$

In this work we are mainly interested in the spatial correlations of paired fermions, and this approximation should not change the main results of our calculations, at least at a qualitative level.

When the dependence on ω of the pairing field is neglected, the anomalous Green’s function is given by the simple expression

$$G^{(12)}(\mathbf{k}, \omega) = -\frac{\Delta(\mathbf{k})}{\omega^2 - \xi^2(k) - \Delta(\mathbf{k})\Delta^*(\mathbf{k})}, \quad (8)$$

with $\xi(k) = k^2/2m_F - \mu_F^*$. By replacing this expression into Eq. (6) and performing the integration over the frequency ω' the following equation for the static limit of the pairing field is obtained

$$\Delta(\mathbf{k}) = \frac{\lambda^2 n_B}{(2\pi)^3} \int d\mathbf{k}' \frac{\epsilon(q)}{[E(\mathbf{k}') + \omega(q)]} \left(\frac{\Delta(\mathbf{k}')}{\omega(q)E(\mathbf{k}')} \right), \quad (9)$$

where $\epsilon(q) = q^2/(2m_B)$, $\omega(q)$ are the excitation energies of the Bose gas calculated within the Bogoliubov approximation, $\omega(q) = \sqrt{(q^2/(2m_B))^2 + 2\gamma n_B q^2/(2m_B)}$, with $\mathbf{q} = \mathbf{k} - \mathbf{k}'$. The quantities $E(\mathbf{k}')$ are the energies of the fermionic quasiparticles, $E(\mathbf{k}') = \sqrt{\xi^2(k') + \Delta(\mathbf{k}')\Delta^*(\mathbf{k}')}$. We notice that the approximation $c_S \gg v_F$ made in Ref. [18], amounts to neglecting the quasiparticle energies $E(\mathbf{k}')$, with respect to the boson energies $\omega(q)$ in the term in square brackets of Eq. (9), whereas the opposite approximation of Ref. [16] is equivalent to assuming $E(\mathbf{k}') \gg \omega(q)$.

From the rotational invariance of the coupling between \mathbf{k} and \mathbf{k}' it ensues that if $\Delta(\mathbf{k})$ is a solution of Eq. (9) also $\Delta(\mathcal{R}\mathbf{k})$, where \mathcal{R} represents any rotation in the \mathbf{k} -space, is a solution. This implies a very high degeneracy for the ground-state energy of the superfluid phase. We will discuss this point below.

The field $\Delta(\mathbf{k})$ can be expanded in spherical harmonics,

$$\Delta(\mathbf{k}) = \sum_{l,m} \sqrt{\frac{4\pi}{2l+1}} \Delta_{l,m}(k) Y_l^m(\Omega_{\mathbf{k}}),$$

with only odd values of l contributing because of the antisymmetric relative-motion wavefunction. In Eq. (9) components of $\Delta(\mathbf{k})$ with different l are coupled in general. We have estimated the relative weight of the components with $l > 1$ using an angle-average approximation for the quasiparticle energy [16]. In this approximation the l -components are decoupled, and in all the cases studied in the present work those with $l > 1$ turn out to be very small with respect to the $l = 1$ component. Hence, we restrict our investigation to the $l = 1$ component alone. Explicit calculations show that this approximation is fairly reliable when we limit ourselves to evaluate average quantities, that is, quantities integrated over the relative momentum of the couple of fermions.

Taking into account only the $l = 1$ component, the pairing field can be put as $\Delta(\mathbf{k}) = \Delta_1(k) \cdot \hat{\mathbf{k}}$, with $\hat{\mathbf{k}} = \mathbf{k}/k$. From Eq. (9) one can easily check that the three-component quantity $\Delta_1(k)$ behaves as a vector. Then, the superfluid state can be identified by this vector. One also can see that if $\Delta_1(k)$ represents a solution of the gap equation (9), also $\mathcal{R}\Delta_1(k)$, where \mathcal{R} denotes an arbitrary rotation in the orbital coordinates, is a solution. This means that only the magnitude $\Delta_1(k)$ can be fixed by the gap equation, whereas the direction of $\Delta_1(k)$ is completely arbitrary. Finally we remark that the energy gap vanishes when \mathbf{k} lies in the plane perpendicular to $\Delta_1(k)$.

The expression for the ground-state energy of the Fermi gas in the superfluid phase can be derived from Eq. (15) of Ref. [16] for the effective action of the pairing field, by adding the contribution of a noninteracting Fermi gas with the effective chemical potential μ_F^* . By exploiting the equation for the pairing field (Eq. (16) of Ref. [16]) and taking the limit of vanishing temperature the energy per fermion in the superfluid phase is given by

$$E_F = \frac{1}{n_F} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{2} \left(\xi(k) - E(\mathbf{k}) + \frac{(\Delta_1 \cdot \hat{\mathbf{k}})^2}{2E(\mathbf{k})} \right) + \mu_F^*. \quad (10)$$

This expression coincides with the usual expression of the BCS theory (see, e. g., Ref. [46]), apart from a factor 1/2 due to the absence of degeneracy for the Fermi gas in the present case.

In order to determine explicitly the field $\Delta_1 \cdot \hat{\mathbf{k}}$ together with the effective chemical potential μ_F^* the equation fixing the fermion density has to be added

$$n_F = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{2} \left(1 - \frac{\xi(k)}{E(\mathbf{k})} \right). \quad (11)$$

Both Eqs. (10) and (11) are invariant under the transformations $\Delta_1(k) \rightarrow \mathcal{R}\Delta_1(k)$. Then, the ground-state energy and the chemical potential of the superfluid phase are determined by the magnitude $\Delta_1(k)$ alone. This implies that the energy of the superfluid ground state is infinitely degenerate. Fermions can test states of the degeneracy subspace through rotations of the vector $\Delta_1(k)$. We notice that in the present case the spins of fermions do not play any role since they are frozen along a fixed direction and there is no coupling between spin and orbital degrees of freedom.

In order to calculate its magnitude we choose a particular direction for the vector $\Delta_1(k)$, say $\Delta_1(k) = (0, 0, \Delta_1(k))$. Moreover, we observe that the scaled quantity, $\tilde{\Delta}_1(k) = \Delta_1(k)/\epsilon_F$ where $\epsilon_F = k_F^2/2m_F = (6\pi^2)n_f^{2/3}/2m_F$ is the Fermi energy, depends only on three dimensionless quantities:

$$b = (\lambda/\gamma)(\lambda n_F/\epsilon_F), \quad c = \xi_B k_F, \quad d = m_B/m_F.$$

The parameter b is determined by the mean field of fermions acting on bosons and by the ratio between the fermion–boson and boson–boson coupling constants, in practice it represents the strength of the effective fermion–fermion interaction. The parameter c is the coherence length of the Bose condensate $\xi_B = 1/\sqrt{2m_B\gamma n_B}$ in units of $1/k_F$, hence it represents the ratio of the range of the effective interaction between fermions to the average interparticle spacing of the Fermi gas. In terms of these dimensionless parameters the equation for $\tilde{\Delta}_1(k)$ reads

$$\tilde{\Delta}_1(\tilde{k}) = \frac{3}{(4\pi)} b \int d\Omega_{\tilde{\mathbf{k}}} \cos(\theta_{\tilde{\mathbf{k}}}) \int \frac{d\tilde{\mathbf{k}}'}{(2\pi)^3} \cos(\theta_{\tilde{\mathbf{k}}'}) \frac{\tilde{q}}{\tilde{E}(\tilde{\mathbf{k}}')cd + \tilde{q}\sqrt{(\tilde{q}c)^2 + 2}} \frac{\tilde{\Delta}_1(\tilde{k}')}{\tilde{E}(\tilde{\mathbf{k}}')\sqrt{(\tilde{q}c)^2 + 2}}, \quad (12)$$

with $\tilde{E}(\tilde{\mathbf{k}}') = E(\mathbf{k}')/\epsilon_F$, $\tilde{\xi}(\tilde{k}) = \xi(k)/\epsilon_F$, and the momenta are expressed in units of the Fermi momentum: $\tilde{k} = k/k_F$ and $\tilde{k}' = k'/k_F$.

III. RESULTS

In this section the properties of the fermionic component of the Bose–Fermi mixture as functions of the two dimensionless parameters b and c are examined and discussed.

The value of the parameter $d = m_B/m_F$ is fixed by specifying the components of the particular system considered: a mixture of ^{87}Rb (bosons) and ^{40}K (fermions) atoms. Once the ratio of the masses is fixed, the ratio between the Fermi and phonon velocities is determined by the value of the parameter c alone, since $v_F/c_S = \sqrt{2}cd$.

Physical quantities of interest will be studied both as a function of the parameter c for a fixed value of b ($b = 5$) and as a function of b for a fixed value of c ($c = 2$). The first

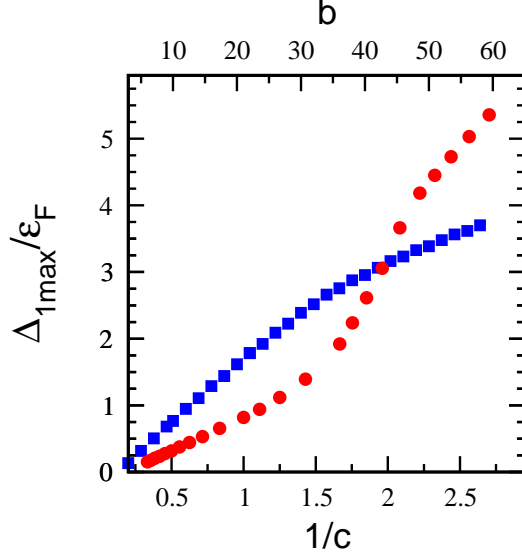


FIG. 1: The maximum value of the p -wave energy gap in units of the Fermi energy as a function of $1/c$ with $b = 5$ (circles), and as function of b with $c = 2$ (squares).

case can be implemented by varying for instance the boson density alone, whereas for the latter the boson–fermion coupling constant can be varied while the other parameters are kept constant. The range of values chosen for the parameters b and c includes domains where the properties of the Fermi gas change drastically. The peculiar behaviour of various physical quantities generally suggests the occurrence in the Fermi gas of a transition from a superfluid long–ranged phase (BCS phase) to a phase of tightly bound pairs of fermions. These composite bosons behave like a weakly interacting Bose condensate (BEC phase). For $b = 5$ and $c = 2$ the obtained energy gap is very small, implying that the fermions are well inside the BCS phase.

It should be remarked that in the considered range of values for c the ratio of the Fermi velocity to the phonon velocity lies in the range $1.2 \lesssim v_F/c_S \lesssim 9$, so the approximations $c_S \gg v_F$ and $c_S \ll v_F$, used in Ref. [18] and in Ref. [16] respectively for deriving the gap equation, are not valid in the region close to the BCS–BEC transition.

In Fig. 1 the maximum value of the magnitude of $\Delta_1(k)$, Δ_{1max} , in units of ϵ_F , is shown both as a function of c and as a function of b . The value of Δ_{1max} increases with increasing b or decreasing c , i.e. when the effectiveness of the fermion–fermion interaction induced by the exchange of phonons increases. A similar behaviour is shown by the value of k at the maximum of $\Delta_1(k)$, i.e. the peak of $\Delta_1(k)$ moves from the Fermi surface towards higher values of k within the range $\simeq (k_F, 4k_F)$. We can also see that Δ_{1max} grows quickly with $1/c$ for $1/c \gtrsim 1.5$, while it shows a smoother behaviour as a function of b instead. Finally, it should be remarked that, as an odd- l component, $\Delta_1(k)$ vanishes for $k \rightarrow 0$, contrary to the case of s -pairing.

Figure 2 shows the behaviour of the effective chemical potential μ_F^* and of the energy per particle E_F for the Fermi component of the mixture as a function of b or $1/c$. For low values of b and/or high values of c the relations $\mu_F^* \simeq \epsilon_F$ and $E_F \simeq 3/5 \epsilon_F$ hold according to the weak coupling BCS theory. When the effectiveness of the interaction between fermions increases, μ_F^* and E_F start to decrease becoming eventually negative. In addition, their values become similar when the fermion–fermion coupling increases. These peculiar features

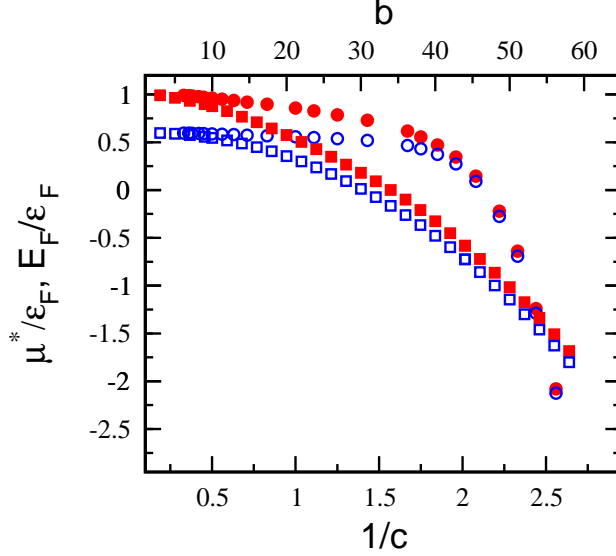


FIG. 2: Effective chemical potential (filled red symbols) and energy per particle (empty blue symbols) of the fermionic gas, in units of the Fermi energy, as a function of $1/c$ with $b = 5$ (circles), and as function of b with $c = 2$ (squares).

indicate the formation of bound pairs of fermions behaving as a condensate Bose gas (BEC phase) [44]. In particular, the difference $\mu_F^* - E_F$, which is related to the interaction between the composite bosons, approaches the values $0.02 \epsilon_F$ for $c = 0.35$ (with $b = 5$) and $0.06 \epsilon_F$ for $b = 60$ (with $c = 2$).

A further remark is required. Equation (9) for the pairing field corresponds to the saddle-point condition for an effective action [16]. Thus, our calculations are essentially performed within the framework of a mean-field theory. The extension of our approach to situations where the energy gap can be larger than the Fermi energy might seem inappropriate. However, analogous studies [39, 45] on the superfluidity in Fermi gases have shown that, provided the temperature is much lower than the critical one, fluctuations of the pairing field about the saddle-point value do not play an important role in determining the energy gap. Hence, we confide that our results can be correct, at least qualitatively.

We turn now to the momentum distribution of the fermions

$$n(\mathbf{k}) = \frac{1}{2} \left(1 - \frac{\xi(k)}{E(\mathbf{k})} \right).$$

The anisotropy of the pairing field changes the features of $n(\mathbf{k})$ significantly with respect to the s -wave superfluidity case. Choosing the direction of $\Delta_1(k)$ along the z -axis the energy gap vanishes for \mathbf{k} lying on the plane $k_z = 0$. Accordingly $n(k_x, k_y, 0)$ is given by the step function $\theta(-\xi(k))$ on the BCS side ($\mu_F^* > 0$) and vanishes on the BEC side ($\mu_F^* < 0$). In the BCS phase the fermion distribution is concentrated across the plane $k_z = 0$. In the BEC phase instead, fermions fill two domains, which are symmetrical with respect to the plane $k_z = 0$. The angle-averaged distribution still shows a critical behaviour when the fermionic component of the mixture crosses the borders between the BCS side and the BEC side. This effect can be observed in Figs. 3 and 4, where the momentum distribution is displayed for two sets of values of b and c , which include both the sides of the transition. On the BCS side the step of the momentum distribution becomes less and less pronounced

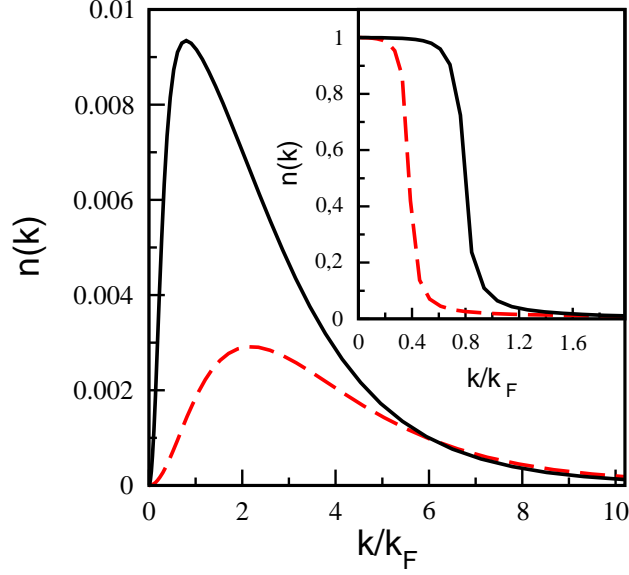


FIG. 3: Main figure: angle-averaged momentum distribution of the fermions, $n(k)$, on the BEC side for two different values of c with fixed $b = 5$: $c = 0.45$, solid line, and $c = 0.4$, dashed line. Inset: same as in main figure but with $c = 0.6$, solid line, and $c = 0.48$, dashed line (BCS side).

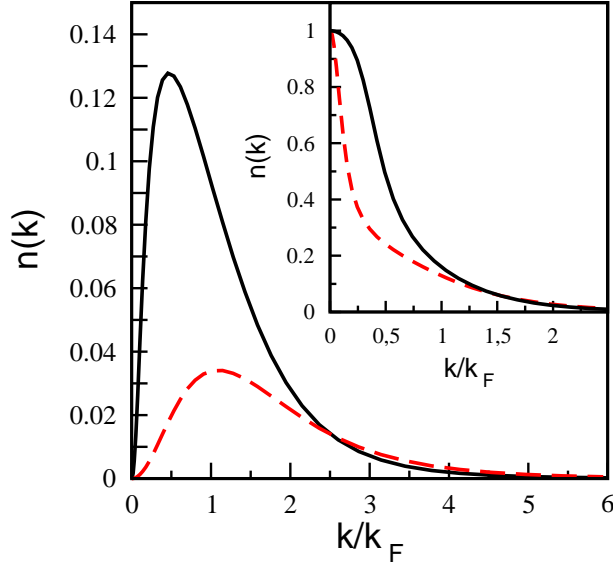


FIG. 4: Main figure: angle-averaged momentum distribution of the fermions, $n(k)$, on the BEC side for two different values of b with fixed $c = 2$: $b = 35$, solid line, and $b = 50$, dashed line. Inset: same as in main figure but with $b = 25$, solid line, and $b = 30$, dashed line (BCS side).

as μ_F^* becomes smaller. On the BEC side the distribution shows the typical behaviour of an odd- l pairing: it vanishes together with $\Delta_1(k)$ at $k \rightarrow 0$ [18, 39]. Moreover its peak becomes less pronounced and moves toward higher value of k/k_F , when the effectiveness of the induced interaction between fermions increases.

Here, we do not discuss the effects of the anisotropy of the pairing field on the quasiparticle spectrum, i.e. a gapless to gapped quantum phase transition, when the effective chemical

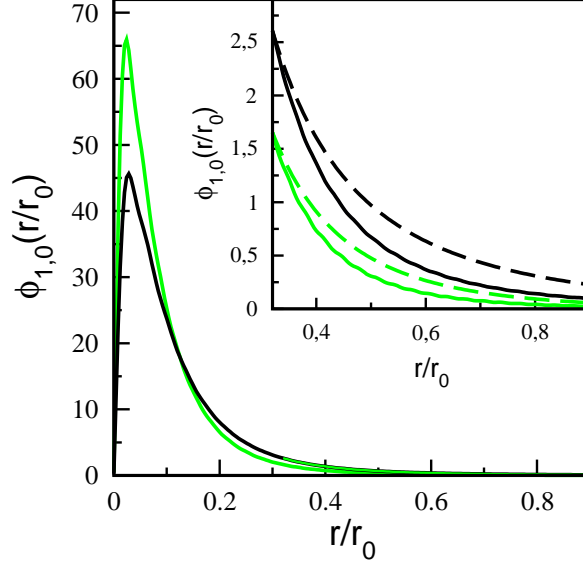


FIG. 5: Main figure: radial wave-function with $l = 1$ of a pair of fermions on the BEC side as a function of the ratio r/r_0 (r_0 is the average spacing of the fermions) for two different values of c : $c = 0.45$ black line and $c = 0.4$ green line, with $b = 5$ in both cases. The wave-functions are normalized according to Eq. (13). Inset: tail of the pair wave-functions (solid lines) and the wave functions of a particle in a square-well potential (dashed lines), see text.

potential vanishes $\mu_F^* = 0$, a topic which has been widely discussed in Refs. [18, 38, 39]. We only observe that in the BCS phase the quasiparticle energy vanishes when \mathbf{k} is on the Fermi surface, $\xi(k) = 0$, and perpendicular to $\Delta_1(k)$.

The transition towards a gas of bound pairs of fermions can be better appreciated by looking at the spatial structure of the pairs. The pair wave function in the center of mass frame is obtained from the Fourier transform of the anomalous density

$$\mathcal{K}_1(\mathbf{k}) = \frac{\Delta_1(k) \cdot \mathbf{k}}{2E(\mathbf{k})}.$$

With our choice for the direction of the vector $\Delta_1(k)$ the pair wave function can be written as

$$\phi(\mathbf{r}) = \sum_l \phi_{l,0}(r) Y_l^0(\Omega)$$

with only odd values of l .

The wave function $\phi(\mathbf{r})$ can contain several partial waves. However, explicit calculations show that the norm of the $l = 1$ component exhausts the norm of $\phi(\mathbf{r})$ within a few percent. Hence, we can neglect the components with $l > 1$.

It is convenient to normalize the radial wave-function $\phi_{1,0}(r/r_0)$ according to the condition

$$\int d\left(\frac{r}{r_0}\right) \left(\frac{r}{r_0}\right)^2 \left| \phi_{1,0}\left(\frac{r}{r_0}\right) \right|^2 = 1, \quad (13)$$

where r_0 is the average spacing of the fermions. On the BCS side the radial wave-function shows the usual damped oscillatory behaviour, where the first peak becomes larger than the secondary oscillations as μ_F^* approaches zero. On the BEC side the oscillations disappear

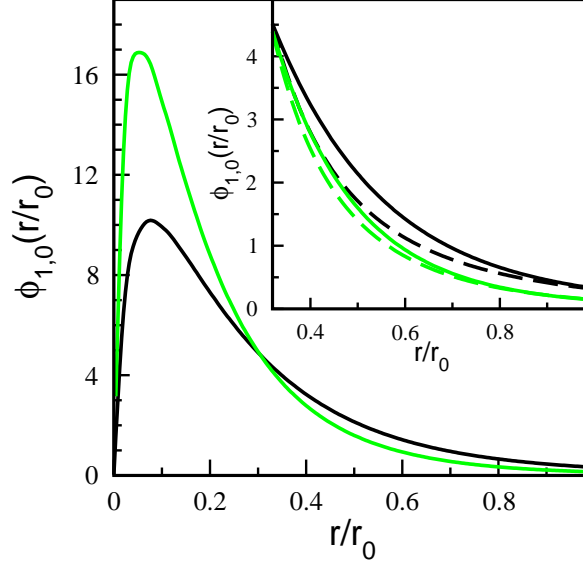


FIG. 6: Same as in Fig. 5 but for $b = 35$ black lines and $b = 50$ green lines, with $c = 2$ in both the cases.

instead, and the wave function acquires a shape similar to that of a bound state with $l = 1$. In Figs. 5 and 6 the radial wave-function is displayed as a function of r/r_0 for two different values of the parameters b and c . We can see that when increasing b and/or decreasing c the wave function is squeezed within a narrower domain about the origin. This simply means that increasing the strength and/or decreasing the range of the induced interaction the binding of a pair of fermions becomes tighter.

The insets of Figs. 5 and 6 show a comparison of the asymptotic behaviour of the radial functions for a pair of fermions with that of the radial wave functions for bound states with $l = 1$ of a particle in a square well potential. For the binding energy and the mass of the bound particle we take the energy and the reduced mass of the pairs of fermions, $BE = 2|E_F|$ and $m_R = m/2$. The point where two wave functions with the same binding energy have been joined, is far enough from the position of the peak of the pair wave-function and is assumed to be outside the potential well. Once more, this comparison suggests the occurrence of a transition of the Fermi component of the mixture toward a Bose gas of tightly bound dimers, when the effectiveness of the induced interaction increases. The root mean square radius of the dimers approaches the values $\simeq 0.2r_0$ for $c = 0.35$ (with $b = 5$) and $\simeq 0.3r_0$ for $b = 60$ (with $c = 2$). This implies that the density of the couples of fermions practically is half that of the Fermi gas.

The calculated values of the difference $\mu_F^* - E_F$ allow us to give an estimate of the ratio between the effective scattering length and the interparticle spacing for the bound pairs of fermions. In the ladder approximation for the self-energy of a dilute Bose gas [46] with repulsive interaction, the relation between the chemical potential and the energy per particle is given by

$$\mu - \frac{E}{N} = \frac{2\pi n_d a_d}{m_d} \left[1 + \frac{64}{5} \left(\frac{n_d a_d^3}{\pi} \right)^{1/2} \right] \quad (14)$$

where n_d , a_d and m_d respectively represent the density, scattering length and mass of the constituents of the Bose gas, with $n_d = n_F/2$, $m_d = 2m_F$ and $\mu - E/N = 2(\mu^* - E_F)$ for

the composite bosons. Dividing Eq. (14) by the Fermi energy one obtains

$$(\mu^* - E_F) \frac{1}{\epsilon_F} = \frac{1}{12^{2/3}} \frac{1}{\pi^{1/3}} a_d n_d^{1/3} \left(1 + \frac{64}{5} \left(\frac{n_d a_d^3}{\pi} \right)^{1/2} \right).$$

With the calculated values for the l.h.s. of the equation above the product $a_d^3 n_d$ approaches the values $1.7 \cdot 10^{-3}$ for $c = 0.35$ (with $b = 5$) and $1.5 \cdot 10^{-2}$ for $b = 60$ (with $c = 2$). These values are sufficiently low to allow us to consider the system of composite bosons as a weakly interacting Bose–Einstein condensate.

The repulsive nature of the force between composite bosons of the BEC phase is consistent with the sign of the difference between the chemical potential and the energy per particle for fermions in a superfluid phase. In fact Eq. (10) clearly shows that anyhow $\mu_F^* - E_F$ is positive. On the other hand a repulsive interaction is required for the stability of the BEC phase against collapse. Our result to some extent is in agreement with the studies of Refs. [48, 49] on the few-body properties of p -wave molecules of fermionic atoms. However, the calculations of Refs. [48, 49] do not rule out the possibility of an attractive interaction between composite particles.

IV. SUMMARY AND CONCLUSIONS

The transition to the BEC regime of a superfluid Fermi gas in a Bose–Fermi mixture can be obtained by tuning the strength of the interatomic interactions via Fano–Feshbach resonances and/or by varying the densities of the two gases. In this paper the relevant quantities for the transition have been expressed as functions of two mutually independent dimensionless parameters b and c . These two parameters are related to the strength and range of the effective interaction between fermions induced by the exchange of one virtual phonon. When the strength of the induced interaction increases and/or its range decreases the fermion pairing evolves from a situation of long-ranged correlations to the onset of tightly bound states. The occurrence of bound states of couples of fermions can be further evinced by the asymptotic behaviour in coordinate space of the anomalous density. This quantity can be interpreted as the relative-motion wave function of the couples of fermions and its asymptotic behaviour approaches that of a particle bound in a spherical potential well. Moreover, when the effectiveness of the induced interaction increases, the chemical potential and the energy per particle of the fermionic component of the mixture approach each other. This behaviour suggests that the pairs of fermions behave as a Bose–Einstein condensate with a weak repulsive interaction between the composite bosons. Finally, the root-mean square radius of the bound pairs is a small fraction of the average interparticle spacing of the fermion gas, so that the gas of composite bosons practically has the same degree of diluteness as the Fermi component of the mixture.

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- [1] A. G. Truscott, K. E. Strecker, W. I. McAlexander, G. B. Partridge, R. G. Hulet, *Science* **291**, 2570 (2001).
 - [2] F. Schreck, L. Khaykovich, K. L. Corwin, G. Ferrari, T. Bourdel, J. Cubizolles, C. Salomon, *Phys. Rev. Lett.* **87**, 080403 (2001).
 - [3] Z. Hadzibabic, C. A. Stan, K. Dieckmann, S. Gupta, M. W. Zwierlein, A. Görlitz, W. Ketterle, *Phys. Rev. Lett.* **88**, 160401 (2002).
 - [4] G. Roati, F. Riboli, G. Modugno, M. Inguscio, *Phys. Rev. Lett.* **89**, 150403 (2002); G. Modugno, G. Roati, F. Riboli, F. Ferlaino, R. J. Brecha, M. Inguscio, *Science* **297**, 2240 (2002).
 - [5] K. Mølmer, *Phys. Rev. Lett.* **80**, 1804 (1998).
 - [6] L. Viverit, C. J. Pethick, H. Smith, *Phys. Rev. A* **61**, 053605 (2000).
 - [7] M. Modugno, F. Ferlaino, F. Riboli, G. Roati, G. Modugno, M. Inguscio, *Phys. Rev. A* **68**, 043626 (2003).
 - [8] S. K. Adhikari, *Phys. Rev. A* **70**, 043617 (2004).
 - [9] B. Ramachandhran, S. G. Bhongale, H. Pu, *Phys. Rev. A* **83**, 033607 (2011).
 - [10] Zeng-Qiang Yu, Shizhong Zhang, Hui Zhai, *Phys. Rev. A* **83**, 041603 (2011).
 - [11] J. Bardeen, G. Baym, D. Pines, *Phys. Rev.* **156**, 207 (1967).
 - [12] M. J. Bijlsma, B. A. Heringa, H. T. C. Stoof, *Phys. Rev. A* **61**, 053601 (2000).
 - [13] H. Heiselberg, C. J. Pethick, H. Smith, L. Viverit, *Phys. Rev. Lett.* **85**, 2418 (2000).
 - [14] D. V. Efremov, L. Viverit, *Phys. Rev. B* **65**, 134519 (2002).
 - [15] L. Viverit, *Phys. Rev. A* **66**, 023605 (2002).
 - [16] F. Matera, *Phys. Rev. A* **68**, 043624 (2003).
 - [17] D.-W. Wang, M. D. Lukin, E. Demler, *Phys. Rev. A* **72**, 051604 (2005); Daw-Wei Wang, *Phys. Rev. Lett.* **96**, 140404 (2006).
 - [18] K. Suzuki, T. Miyakawa, T. Suzuki, *Phys. Rev. A* **77**, 043629 (2008).
 - [19] K. Maeda, G. Baym, T. Hatsuda, *Phys. Rev. Lett.* **103**, 085301 (2009).
 - [20] C. A. Stan, M. W. Zwierlein, C. H. Schunck, S. M. F. Raupach, W. Ketterle, *Phys. Rev. Lett.* **93**, 143001 (2004).
 - [21] S. Inouye, J. Goldwin, M. L. Olsen, C. Ticknor, J. L. Bohn, D. S. Jin, *Phys. Rev. Lett.* **93**, 183201 (2004).
 - [22] F. Ferlaino, C. D’Errico, G. Roati, M. Zaccanti, M. Inguscio, G. Modugno, A. Simoni, *Phys. Rev. A* **73**, 040702(R) (2006).
 - [23] B. Deh, C. Marzok, C. Zimmermann, P. W. Courteille, *Phys. Rev. A* **77**, 010701(R) (2008).
 - [24] C. Ospelkaus, S. Ospelkaus, K. Sengstock, K. Bongs, *Phys. Rev. Lett.* **96**, 020401 (2006); S. Ospelkaus, C. Ospelkaus, L. Humbert, K. Sengstock, K. Bongs, *ibid.* **97**, 120403 (2006).
 - [25] M. Zaccanti, C. D’Errico, F. Ferlaino, G. Roati, M. Inguscio, G. Modugno, *Phys. Rev. A* **74**, 041605(R) (2006).
 - [26] G. Modugno, in *Proceedings of the International School of Physics Enrico Fermi—Course CLXIV*, edited by M. Inguscio, W. Ketterle and C. Salomon (IOS Press, Amsterdam, 2008).
 - [27] J. J. Zirbel, K.-K. Ni, S. Ospelkaus, J. P. D’Incao, C. E. Wieman, J. Ye, D. S. Jin, *Phys.*

- Rev. Lett. **100**, 143201 (2008).
- [28] M. Yu. Kagan, I. V. Brodsky, D. V. Efremov, A. V. Klapptsov, Phys. Rev. A **70**, 023607 (2004).
 - [29] A. Storozhenko, P. Schuck, T. Suzuki, H. Yabu, J. Dukelsky, Phys. Rev. A **71**, 063617 (2005).
 - [30] S. Powell, S. Sachdev, H. P. Büchler, Phys. Rev. B **72**, 024534 (2005).
 - [31] A. V. Avdeenkov, D. C. E. Bortolotti, J. L. Bohn, Phys. Rev. A **74**, 012709 (2006).
 - [32] L. Salasnich, F. Toigo, Phys. Rev. A **75**, 013623 (2007).
 - [33] T. Nishimura, A. Matsumoto, H. Yabu, Phys. Rev. A **77**, 063612 (2008).
 - [34] T. Watanabe, T. Suzuki, P. Schuck, Phys. Rev. A **78**, 033601 (2008).
 - [35] F. M. Marchetti, C. J. M. Mathy, D. A. Huse, M. M. Parish, Phys. Rev. B **78**, 134517 (2008).
 - [36] E. Fratini, P. Pieri, Phys. Rev. A **81**, 051605(R) (2010).
 - [37] I. Bloch, J. Dalibard, W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008); S. Giorgini, L. P. Pitaevskii, S. Stringari, *ibid.* **80**, 1215 (2008); and references therein.
 - [38] V. Gurarie, L. Radzihovsky, A. V. Andreev, Phys. Rev. Lett. **94**, 230403 (2005).
 - [39] M. Iskin, C. A. R. Sa' de Melo, Phys. Rev. Lett. **96**, 040402 (2006); Phys. Rev. A **74**, 013608 (2006).
 - [40] M. Baldo, U. Lombardo, P. Schuck, Phys. Rev. C **52**, 975 (1995).
 - [41] U. Lombardo, P. Nozières, P. Schuck, H.-J. Schulze, A. Sedrakian, Phys. Rev. C **64**, 064314 (2001).
 - [42] A. A. Isayev, Phys. Rev. C **78**, 014306 (2008).
 - [43] B. Y. Sun, H. Toki, J. Meng, Phys. Lett. B **683**, 134 (2010).
 - [44] A. J. Legget, *Quantum Liquids: Bose condensation and Cooper pairing in condensed-matter systems* (Oxford University Press, Oxford, 2006).
 - [45] J. R. Engelbrecht, M. Randeria, C. A. R. Sa' de Melo, Phys. Rev. B **55**, 15153 (1997).
 - [46] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).
 - [47] A. I. Larkin, Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **47**, 1136 (1964) [Sov. Phys. JEPT **20**, 762 (1965)]; P. Fulde, R. A. Ferrell, Phys. Rev. **135**, A550 (1964).
 - [48] M. Jona-Lasinio, L. Pricoupenko, Y. Castin, Phys. Rev. A **77**, 043611 (2008).
 - [49] J. P. D'Incao, B. D. Esry, C. H. Greene, Phys. Rev. A **77**, 052709 (2008).